OCR

Oxford Cambridge and RSA

Monday 16 May 2022

AS Level Further Mathematics B (MEI)

Y410/01 Core Pure

Worked Solutions

Printed Answer Booklet

Time allowed: 1 hour 15 minutes

You must have:
- Question Paper Y410/01 (inside this document)
- the Formulae Booklet for Further Mathematics B (MEI)
- a scientific or graphical calculator

R red level
- longer questions (6+ marks)
- higher level problem solving
- harder A Level content

A amber level
- Shorter questions (3 - 6 marks)
- low level problem solving
- harder AS / easier A Level content.

G green level
- Short questions (1 - 3 marks)
- minimal problem solving
- AS / easier A Level content.

E explanation

Q1: Matrices

Q2: Vectors

Q3: Complex Numbers

Q4: Algebra

Q5: Complex Numbers

Q6: Algebra, Proof

Q7: Complex Numbers

Q8: Matrices

Grade Boundaries

Grade	A	B	C	D	E	U
Mark / 60	38	32	26	20	15	0

ADVICE
- Read each question carefully before you start your answer.

G

1 **(a)** **(i)** Write the following simultaneous equations as a matrix equation.

$$x+y+2z=7$$
$$2x-4y-3z=-5$$
$$-5x+3y+5z=13$$

[1]

Writing in matrix form,

$$\begin{pmatrix} 1 & 1 & 2 \\ 2 & -4 & -3 \\ -5 & 3 & 5 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 7 \\ -5 \\ 13 \end{pmatrix}$$

G

(ii) Hence solve the equations. **[2]**

To solve we multiply both sides by the inverse of our coefficient matrix (found using calculator)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = -\frac{1}{34}\begin{pmatrix} -11 & 1 & 5 \\ 5 & 15 & 7 \\ -14 & -8 & -6 \end{pmatrix}\begin{pmatrix} 7 \\ -5 \\ 13 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -3/2 \\ 4 \end{pmatrix}$$

hence $x=\frac{1}{2}$, $y=-\frac{3}{2}$, $z=4$

A

(b) Determine the set of values of the constant k for which the matrix equation

$$\underbrace{\begin{pmatrix} k+1 & 1 \\ 2 & k \end{pmatrix}}_{M}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 23 \\ -17 \end{pmatrix}$$

has a unique solution. **[3]**

The system only has a unique solution if the coefficient matrix has an inverse, so $\det M \neq 0$.

First we find $\det M$.

$$\det M = k(k+1)-2$$
$$= k^2+k-2$$

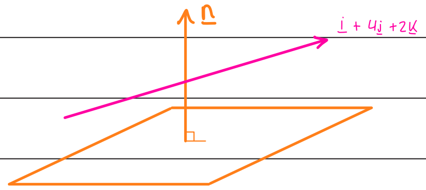
Now we find the values of k such that $\det M = 0$.

$$k^2+k-2=0$$
$$(k+2)(k-1)=0$$
$$k=-2,\ k=1$$

Hence the system has a unique solution if $k \neq -2$ or $k \neq 1$.

A

2 (a) Show that the vector $\mathbf{i}+4\mathbf{j}+2\mathbf{k}$ is parallel to the plane $2x+y-3z=10$. **[3]**



So for the plane to be parallel to a vector, the vector and normal must be perpendicular.

Recall that if $\underline{a}\cdot\underline{b}=0$, $\underline{a}$ and $\underline{b}$ are perpendicular. So

$$1(2)+4(1)+2(-3)$$
$$=2+4-6=0$$

Hence since the vector and the normal are perpendicular, the plane and vector are parallel.

G

(b) Determine the acute angle between the planes $2x+y-3z=10$ and $x-y-3z=3$. **[4]**

Recall that $\cos\theta=\dfrac{\underline{a}\cdot\underline{b}}{|\underline{a}||\underline{b}|}$.

So we use this to find the angle between the normals of the planes, which is also the angle between the planes.

$$\underline{a}\cdot\underline{b}=2(1)+1(-1)-3(-3)=10$$

So $\cos\theta=\dfrac{10}{\sqrt{2^2+1^2+3^2}\sqrt{1^2+1^2+3^2}}=\dfrac{10}{\sqrt{154}}$

hence $\theta=\arccos\left(\dfrac{10}{\sqrt{154}}\right)=36.310\ldots$
$=36.3°$

A

3 The complex number z satisfies the equation $5(z-\mathrm{i}) = (-1+2\mathrm{i})z^*$.

Determine z, giving your answer in the form $a+b\mathrm{i}$, where a and b are real. **[5]**

We can solve this by substituting $z = a + bi$ into the equation.

z^* is the complex conjugate

let $z = a + bi$.

so $5(a+bi-i) = (-1+2i)(a-bi)$

$5a + 5bi - 5i = -a + bi + 2ai - 2bi^2$

$5a + (5b-5)i = -a + bi + 2ai + 2b$

$5a + (5b-5)i = (2b-a) + (2a+b)i$

Now we equate the real and imaginary parts.

Real: $5a = 2b - a \Rightarrow b = 3a$

Imaginary: $5b - 5 = 2a + b$

$5(3a) - 5 = 2a + 3a$

$10a = 5$

$a = \frac{1}{2}$, so $b = \frac{3}{2}$

solve for a and b

Hence $z = \frac{1}{2} + \frac{3}{2}i$ ← write in required form

A

4 **In this question you must show detailed reasoning.**

The equation $z^3 + 2z^2 + kz + 3 = 0$, where k is a constant, has roots α, $\frac{1}{\alpha}$ and β.

Determine the roots in exact form. **[6]**

We can find the roots by considering $\sum \alpha$ and $\sum \alpha\beta\gamma$.

Using $\sum \alpha = -\frac{b}{a}$ (sum of roots)

$\alpha + \frac{1}{\alpha} + \beta = -\frac{2}{1}$

$\alpha + \frac{1}{\alpha} + \beta = -2$

$\alpha^2 + 1 + \alpha\beta + 2\alpha = 0$

$\alpha^2 + \alpha\beta + 2\alpha + 1 = 0$ (A)

Using $\sum \alpha\beta\gamma = -\frac{d}{a}$

$\alpha \times \frac{1}{\alpha} \times \beta = -\frac{3}{1}$

$\beta = -3$ (B)

Substituting (B) into (A), $\alpha^2 + \alpha(-3) + 2\alpha + 1 = 0$

$\alpha^2 - \alpha + 1 = 0$

$\alpha = \dfrac{1 \pm \sqrt{(-1)^2 - 4 \times 1 \times 1}}{2 \times 1}$

$\alpha = \frac{1}{2} \pm \frac{\sqrt{-3}}{2}$

$\alpha = \frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

If $\alpha = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{1}{\alpha} = \frac{1}{2} - \frac{\sqrt{3}}{2}i$

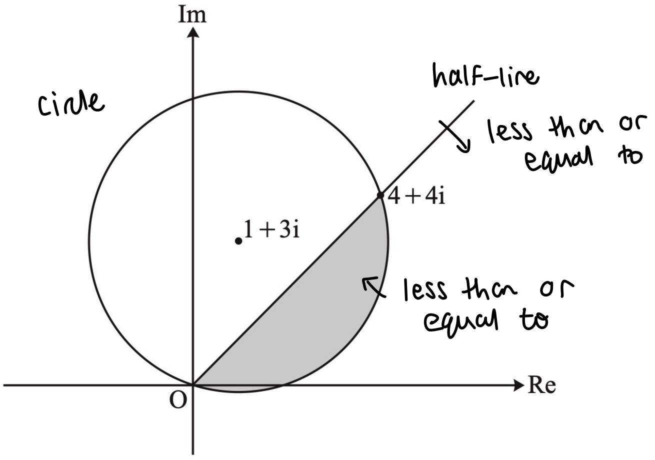
If $\alpha = \frac{1}{2} - \frac{\sqrt{3}}{2}i$, $\frac{1}{\alpha} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$

Check our roots are always α, $\frac{1}{\alpha}$ and β.

Hence the roots are -3, $\frac{1}{2} + \frac{\sqrt{3}}{2}i$, $\frac{1}{2} - \frac{\sqrt{3}}{2}i$

5 An Argand diagram is shown below. The circle has centre at the point representing $1+3\text{i}$, and the half line intersects the circle at the origin and at the point representing $4+4\text{i}$.

A



State the **two** conditions that define the set of complex numbers represented by points in the shaded segment, including its boundaries. **[5]**

circle: radius = distance between $1+3i$ and $4+4i$

$= \sqrt{(4-1)^2 + (4-3)^2} = \sqrt{10}$

hence $|z-(1+3i)| \leq \sqrt{10}$

$|z-1-3i| \leq \sqrt{10}$

half-line: angle = $\arctan\left(\frac{4}{4}\right) = \arctan 1 = \frac{\pi}{4}$

hence $\arg(z) \leq \frac{\pi}{4}$

G

6 (a) Using standard summation formulae, show that $\sum_{r=1}^{n} r(r+2) = \frac{1}{6}n(n+1)(2n+7)$. **[4]**

First we expand out the expression we are summing.

$\sum_{r=1}^{n} r(r+2) = \sum_{r=1}^{n} r^2 + 2r$

From the formula book we use $\sum_{r=1}^{n} r^2 = \frac{1}{6}n(n+1)(2n+1)$ and recall that $\sum_{r=1}^{n} r = \frac{1}{2}n(n+1)$. So

$= \frac{1}{6}n(n+1)(2n+1) + 2\left(\frac{1}{2}n(n+1)\right)$

$= \frac{1}{6}n(n+1)(2n+1) + n(n+1)$

$= \frac{1}{6}n(n+1)\left[(2n+1) + 6\right]$

$= \frac{1}{6}n(n+1)(2n+7)$ as required

A

(b) Use induction to prove the result in part **(a)**. **[6]**

Step one: base case

When $n=1$, $\sum_{r=1}^{1} r(r+2) = 1(1+2) = 3$

$\sum_{r=1}^{n} r(r+2) = \frac{1}{6}(1)(1+1)(2+7)$

$= \frac{1}{6} \times 2 \times 9 = 3 \quad \therefore$ true for $n=1$.

Step two: assumption

Assume true for $n=k$, so $\sum_{r=1}^{k} r(r+2) = \frac{1}{6}k(k+1)(2k+7)$

Step three: inductive step

Using the assumed result for $n=k$,

$$\sum_{r=1}^{k+1} r(r+2) = \sum_{r=1}^{k} r(r+2) + (k+1)(k+1+2)$$

$$= \frac{1}{6}k(k+1)(2k+7) + (k+1)(k+3)$$

$$= \frac{1}{6}(k+1)\left[k(2k+7) + 6(k+3)\right]$$

$$= \frac{1}{6}(k+1)(2k^2 + 7k + 6k + 18)$$

$$= \frac{1}{6}(k+1)(2k^2 + 13k + 18)$$

$$= \frac{1}{6}(k+1)(2k+9)(k+2)$$

$$= \frac{1}{6}(k+1)((k+1)+1)(2(k+1)+7)$$

$\therefore$ true for $n=k+1$.

Factor and Simplify

Step four: conclusion

If the result is true for $n=k$, it is true for $n=k+1$. Since it is true for $n=1$, it is true for all positive integer values of n.

A

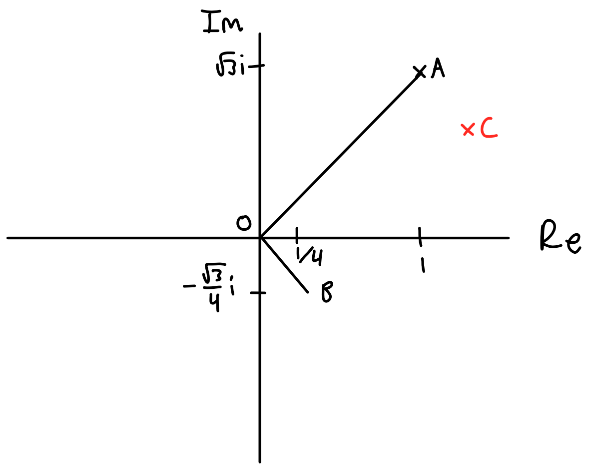
7 On an Argand diagram, the point A represents the complex number z with modulus 2 and argument $\frac{1}{3}\pi$. The point B represents $\dfrac{1}{z}$.

(a) Sketch an Argand diagram showing the origin O and the points A and B. **[2]**

$$\text{So } z = 2\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right) = 1 + \sqrt{3}i$$

$$\frac{1}{z} = \frac{1}{1+\sqrt{3}i} \times \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{1}{4} - \frac{\sqrt{3}}{4}i$$

Find z and $\frac{1}{z}$



Be sure to label axes and points.

R

(b) The point C is such that OACB is a parallelogram. C represents the complex number w.

Determine each of the following.

- The modulus of w, giving your answer in exact form.
- The argument of w, giving your answer correct to **3** significant figures. **[7]**

We can show C on the diagram in part a).

$$\text{So } \vec{OC} = \vec{OA} + \vec{AC}$$

$$w = z + \frac{1}{z}$$

these two expressions are equivalent

$$= 1 + \sqrt{3}i + \frac{1}{4} - \frac{\sqrt{3}}{4}i$$

$$= \frac{5}{4} + \frac{3\sqrt{3}}{4}i$$

$$\text{hence } |w| = \sqrt{\left(\frac{5}{4}\right)^2 + \left(\frac{3\sqrt{3}}{4}\right)^2} = \frac{\sqrt{13}}{2}$$

$$\arg w = \arctan\left(\frac{3\sqrt{3}/4}{5/4}\right) = \arctan\frac{3\sqrt{3}}{5}$$

$$= 0.8046\ldots$$

$$= 0.805$$

So modulus of w is $\frac{\sqrt{13}}{2}$ and argument of w is 0.805 rad

(answer space continued on next page)

7(b)	(continued)

A

8 A transformation T of the plane has matrix **M**, where $\mathbf{M} = \begin{pmatrix} \cos\theta & 2\cos\theta - \sin\theta \\ \sin\theta & 2\sin\theta + \cos\theta \end{pmatrix}$.

(a) Show that T leaves areas unchanged for all values of θ. **[2]**

First we find det M

$\det M = \cos\theta\,(2\sin\theta + \cos\theta)$
$- \sin\theta\,(2\cos\theta - \sin\theta)$
$= 2\sin\theta\cos\theta + \cos^2\theta - 2\sin\theta\cos\theta + \sin^2\theta$
$= \sin^2\theta + \cos^2\theta$
$= 1$ using AS Trigonometry

Hence since det M = 1, T preserves area.

R

(b) Find the value of θ, where $0 < \theta < \frac{1}{2}\pi$, for which the y-axis is an invariant line of T. **[4]**

Here we set up $M\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y' \end{pmatrix}$

$\begin{pmatrix} \cos\theta & 2\cos\theta - \sin\theta \\ \sin\theta & 2\sin\theta + \cos\theta \end{pmatrix}\begin{pmatrix} 0 \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ y' \end{pmatrix}$

$(2\cos\theta - \sin\theta)\,y = 0$

so $2\cos\theta - \sin\theta = 0$ for this to be true

$2 - \tan\theta = 0$

$\theta = \arctan 2 = 1.1071\ldots = 1.11$ rad

A

The matrix **N** is $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(c) **(i)** Find $\mathbf{MN}^{-1}$. **[2]**

First find N^{-1}

$\det N = 1 - 2(0) = 1$

$N^{-1} = \frac{1}{1}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

Now find product MN^{-1}.

$MN^{-1} = \begin{pmatrix} \cos\theta & 2\cos\theta - \sin\theta \\ \sin\theta & 2\sin\theta + \cos\theta \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} \cos\theta & -2\cos\theta + 2\cos\theta - \sin\theta \\ \sin\theta & -2\sin\theta + 2\sin\theta + \cos\theta \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

A

(ii) Hence describe fully a sequence of two transformations of the plane that is equivalent to T. **[4]**

MN^{-1} is a general rotation matrix of $\theta°$ ACW about O.

So if $MN^{-1} = R$

$MN^{-1}N = RN$

$M = RN$

Hence T is a N then R.

So T is ① Shear, x-axis fixed, with $(0, 1)$ mapped to $(2, 1)$.

② rotation by $\theta°$ anti-clockwise about the origin.

ADDITIONAL ANSWER SPACE

If additional space is required, you should use the following lined page(s). The question number(s) must be clearly shown in the margin(s).